CHAPTER

4

Determining What Children Know: Dynamic versus Static Assessment

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You cannot step twice into the same stream. For as you are stepping in, other waters are ever flowing.

—Heraclitus

ASSESSMENT should guide teaching. It should be continuous and provide information about the "zone of proximal development" (Vygotsky 1978). To do so, it needs to foresee where and how one can anticipate that which is just coming into view in the distance (Streefland 1985). It needs to capture genuine mathematizing: children's strategies, their ways of modeling realistic problems, and their understanding of key mathematical ideas. Bottom line, it needs to capture where the child is on the landscape of learning—where she has been, what her struggles are, and where she is going: it must be *dynamic* (Fosnot and Dolk 2001; van den Heuvel-Panhuizen 1996).

Most forms of assessment are not dynamic or continuous; they are *static* and discrete, designed to determine what a child cannot do rather than what he can do (de Lange 1992). They usually assess skills (and the deficit of them) rather than the landscape of learning. A medical model of teaching and learning underlies these practices. The expressed intent is to identify problems and label them in order to prescribe treatment. Remediation (in contrast to intervention and prevention) then becomes the goal.

Because intervention is designed to prevent problems, assessments must give teachers the information needed to offer structures and content that will ensure success. Teachers can carry out dynamic assessment both in the moment and formally. This chapter presents many examples.

Listening in the Moment

> When I'm looking at the student work afterward, that's helpful. But a lot of times I miss the critical thing because the student work is the product, kind of what happened at the end. And although they [the children] sometimes will explain their thinking, and I can figure out what they've done on paper, I get more information from the process by being there in the moment. (Michael Flynn [Storeygard 2009, p. 97])

Flynn's statement illustrates one way that teachers do ongoing assessment with their students: by listening to their students as they work on mathematics, both individually and as a part of large- and small-group discussions. Careful listening is powerful in informing teachers how to adjust their teaching.

Listening in the moment is complex. Teachers do not have the luxury of a rewind button, and they often are listening to a student or group with multiple activities and conversations going on around them. To know what to pay attention to, teachers must have a clear idea of the developmental pathways on the landscape and the mathematical goals of the activity they are doing, and using this knowledge they must be able to make sense of what a child is saying. To bring out student thinking and to then support development, teachers must know what to celebrate (because it is a landmark step forward) and how to challenge students to support further development. Their questions need to fit the situation and provide assessment information, and their work needs to be within the zone of proximal development (Vygotsky 1978). Meeting these criteria is no easy task.

Listening for strengths and celebrating them

For teachers, noticing what struggling students do not know is often easier than recognizing what they do know. Teachers must find students' strengths to build their self-confidence but also find entry points to help them build understanding. One first-year teacher noted, "Plagued by my own frustrations over how to help Tamara, I had been fixating on what she didn't know and it had come to seem this was everything. Using end-of-unit tests as the only form of assessment was dangerous in the same way because I was highlighting only her areas of need and then feeling overwhelmed by the amount" (Storeygard 2009, p. 70).

By listening to Tamara, and recording what she did know, the teacher learned that Tamara actually had constructed some of the big ideas related to early numeracy, and these accomplishments were to be celebrated. She could count verbally, use one-to-one correspondence with objects, and recognize that the last number word in her count told "how many" were in the set: she understood cardinality. She could also compare the magnitude of numbers by using number cards (zero through ten) and move pieces along a track meaningfully when playing board games.

Once the teacher found that Tamara's understanding of zero to ten was solidifying, she could then plan the next steps to take. She might focus on finding numbers within others; for example, five and three are within eight. To do this she could begin by using a board game with tracks colored in alternating groups of fives and marked with the decades: 10, 20, 30, and so forth. (See fig. 4.1.) If one die (in the pair of game dice) has only the numeral 5 on each face instead of pips (or dots), and the other is a normal die with one to six pips on the faces, at a given roll the pair of dice might read five plus four pips or five plus three pips. This game would support Tamara to count on from five instead of counting by ones, and the game board would help her recognize when she landed on a multiple of ten.

Leapfrog game board

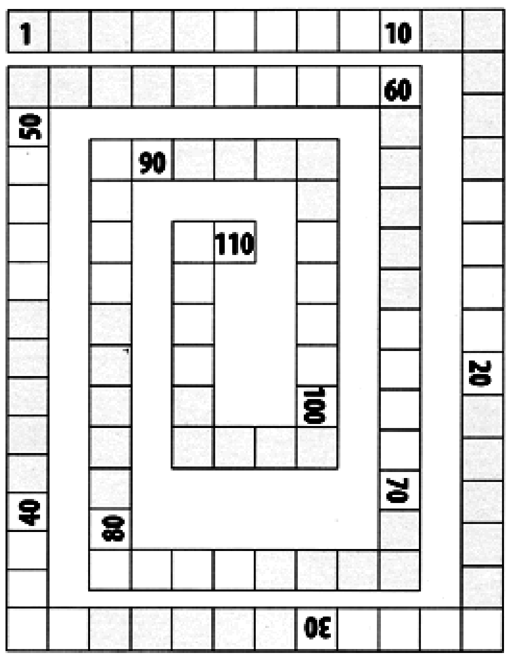

Fig. 4.1. Game board with labeled decades (reproduced from Catherine Twomey Fosnot, "Measuring for the Art Show," p. 55, a unit in *Contexts for Learning Mathematics*, Portsmouth, N.H.: Firsthand Heinemann, 2007)

Once Tamara can use fives and tens, she will probably make progress learning the basic addition facts. For example, if she can consider 6 as 5 + 1 and knows 5 + 5, she will be able to envision 5 + 6 as 5 + 5 + 1. Once Tamara understands the teen numbers (e.g., that 17 is composed of 10 + 7), the teacher might use the 10+ facts to help Tamara figure out facts that have 9 as one addend. For the problem 9 + 7, the teacher could help Tamara envision it as one less than 10 + 7. Decisions like these are based on a deep understanding of the development of number sense, not on a list of prescribed behavioral objectives.

Listening for developmental landmarks and posing questions

Thinking in broad topic terms, such as "Jessica can't count" or "Steven doesn't understand multiplication," is usually not helpful. Instead, by observing and listening carefully as students work, teachers can find out more specifically where students are on the landscape of learning for that topic. "When Jessica counts, where does she get stuck? Does she use the 1–9 sequence between the decades? Is she counting one number per object? How does she keep track of her count? Does she know that the number she ends on is a value and that the numbers in the counting sequence grow in a plus-one fashion?"

Dynamic assessment also requires teachers to think about whether a student seems to understand a concept in some situations but not in others. Considering questions such as the following is helpful: "Does Steven understand multiplication across representations and contexts, or does his understanding break down when new models such as arrays or ratio tables are introduced? Does he realize that repeated additions can be regrouped, for example, that one can think of four groups of eight as two groups of sixteen? Can he see smaller partial products inside larger ones and then use them? Can he multiply by ten and use this partial product?" Teachers learn how to ask carefully crafted questions like these as they learn more about the mathematics they are teaching, the sequence of how mathematical ideas and strategies develop, the typical range of responses they might expect, and the particular mathematical strengths and challenges of each student.

Sometimes beginning teachers want a standard list of questions to ask in order to help them find out about students' thinking. General questions like "Can you tell me more about this?" and "How do you know?" may be useful in getting more information, but as they gain experience in questioning, teachers realize that the best questions usually are those that come from understanding math development and observing and listening in the moment. They are not planned.

The following excerpt is from a second-grade class in which students were rolling dice and moving the corresponding number of spaces on a 100 chart. The teacher is questioning a student having difficulty (Storeygard 2009, DVD).

Teacher: So let's do one together just for fun. Let's pretend you're at 20 and you roll 20. Where would you move?

Child: Umm, 50.

Teacher: Show me.

Child: No, 40.

Teacher: Show me why it would be 40.

Child: No, it's 50.

Teacher: Show me why it would be 50.

Child: Umm, ok. [*He crosses out 25 with his pencil.*]

Teacher: Can you talk out loud? How much would that be? [*He points to the 25.*]

Child: 25.

Teacher: But how much did you move when you went from 20 to 25?

Child: 1, 2, 3, 4, 5 … 5. [*He draws an* X *on 30.*] Another 5 … 30. Umm … 30, 45, 50. It's not 20 moves. It's 40.

Teacher: Show me with the piece what it looks like as you make a jump of 20.

Child: OK. [*He jumps on from 20, two rows, to 40.*]

Teacher: OK, so why is that 20? Can you show me?

Child: There's 10 in each row … and 10, plus 10, is 20.

Teacher: Can I ask you one more thing? If you went from 20 to 40 in jumps of 5s, what would that look like?

Child: [*He starts his piece at 20 and moves the piece one space as he says each number.*] 1, 2, 3, 4, 5 … 1, 2, 3, 4, 5 … 1, 2, 3, 4, 5 … 1, 2, 3, 4, 5.

Teacher: So you did it two ways! You did two jumps of tens and landed on 40, and then you made four jumps of fives and landed on 40.

The teacher did not tell the child he was incorrect or say something like "Are you sure?" He set up a situation in which he tried to help the child think through what he was doing. In the process, he learned what the child understood and where he needed more support. In his journal, the teacher reflected,

> I do this often with students. Early on in my career I discovered I would lead kids into telling me the right answer based on the types of questions I asked. The kids could tell whether their answers were wrong based on the type of response I gave. So now (right or wrong) I try to respond the same way so they have to think on their own.

Once a teacher has gathered information about how a child is thinking and what the areas of confusion are, she might pose a question to help that student access and build on prior knowledge. For example, a second-grade teacher was helping a child, Sylvia, find the sum of an expression with many addends: 10 + 5 + 7 + 25 + 3 + 8 + 20 + 2. Before this activity, the child had been playing Tens Go Fish (Russell et al. 2008a), a card game that involved making combinations to ten—and the teacher had noticed that the child knew many of the combinations of ten. Using this information, the teacher asked, "Sylvia, pretend these numbers are the cards you have to play Tens Go Fish. Would you be able to make any combinations of 10?" Right away the child recognized that she could combine 7 and 3, and 8 and 2. The teacher celebrated her answer and then challenged Sylvia by asking, "I wonder if there are other friendly numbers that would be helpful to make besides ten?" Sylvia pondered the question and then with a broad smile said, "I can make 30 with the 5 and the 25, and with the 10 and 20" (Vaisenstein 2009, p. 57).

Assessing during group work

Circulating as students work in pairs or groups, teachers often arrive in the middle of an activity. Too often they immediately ask children to explain what they are doing. Doing so may not only be distractive but may also cause teachers to miss wonderful moments for assessment. Listening carefully first is usually more helpful, both to find out how students are thinking and to observe how they are interacting. Let us listen in on another conversation. This one takes place in Greg's fourth-grade classroom. Greg had recently come to appreciate that his students needed to grapple with genuinely problematic situations to allow them to make sense of big mathematical ideas. We are joining the class in the middle of their first big investigation. They are working with the *Contexts for Learning Mathematics* unit *Muffles Truffles* (Cameron and Fosnot 2007). The children are paired and are hard at work creating large assortment boxes for truffles from smaller 2 × 5 boxes (each of which holds ten truffles of a specific type).

Malik is a student whom Greg is concerned about. Malik does not know many of his multiplication facts automatically yet, and although he can model multiplicative situations correctly, he usually does so inefficiently, by drawing rows and rows of tally marks and counting them by ones. Today Greg paired Malik with Jerome. The purpose of the investigation is to build arrays with smaller partial products (the 2 × 5 boxes) as a way to explore several big ideas related to the associative and distributive properties. Although Greg is committed to letting his students grapple and make sense of problems themselves, he worries that these complex ideas may be too difficult for Malik. Greg notices from across the room that the boy has already covered his paper with pictures of chocolate boxes with every square drawn in, and Greg worries that drawing each square will be an even more time-consuming version of the tally marks. He heads across the room to intervene. As he approaches the pair, he hears the following conversation.

Jerome is pointing to one of the arrays and speaking to Malik in an agitated voice. "I don't think this is right," he declares.

Malik responds in an equally agitated tone, "I know that it's right. I counted every single square and there are forty squares. I made four little candy boxes and each little box is a two by five, and that's ten squares. There are four boxes with ten squares, so that's forty … ten, twenty, thirty, forty."

Greg is surprised to hear Malik counting by tens. Malik is not counting by ones as Greg had feared. He wants to hear more of Malik's thinking, but Malik is glaring at Jerome, so Greg decides to defuse the tension first by validating Malik's thinking. "Malik, that sounds very interesting. Let's see if Jerome and I understood what you were saying. Jerome, can you repeat in your own words what Malik said?"

"He said that this drawing is right because it has 40 squares. He knows that the small box has ten, and four times ten equals forty … so that has to be right," Jerome responds.

"Does that sound about right, Malik?" Greg asks, and Malik nods grudgingly. Greg asks the boys for further clarification. "Well now I'm confused. If you both agree that Malik is correct, then what is wrong with the drawing?"

"It has too many squares," Jerome declares. In his mind, drawing the outline of the smaller boxes is sufficient.

"No it doesn't! It has to have 40 squares, or how will people know that it's right?" Malik says in defense of his representation. "It's 2 by 5, plus 2 by 5, plus 2 by 5, plus 2 by 5, and how can I show that without all the squares?"

The conversation continues, and together the boys eventually decide that they will draw only the outlines of the smaller boxes and label each with the equation $2 \times 5 = 10$. Interestingly, after Malik is finally convinced that his picture does not have to show all the squares to prove he is correct, he is pleased with their new representation and seemingly relieved because it will take much less time than drawing all the little squares. Greg leaves the boys to create more boxes with their new representation. As he leaves, Malik shouts out in excitement, "Hey look, that's four tens, but it's also eight fives. That could be a different box! Oh no, wait a minute, maybe not." Greg pauses a moment but resists the temptation to turn back to the boys' table. He decides that he will let the boys work together to sort out the question themselves and check back in a few minutes to see how they are doing.

In his journal that evening, Greg writes,

> I'm impressed with what Malik explained today and puzzled about how he could make sense of so many complicated things and yet do so poorly on his classwork and the end-of-unit assessments I've been giving. Pondering their argument, I now realize that their discussion was not about how to solve the problem but about how to show their thinking to others. This was powerful in helping Malik. This episode also helps me understand the importance of having students spend most of their time in pairs solving problems and writing up their solutions for an audience. In the past I gave such little time for talking and relied on written work to judge what my kids knew. Maybe I've been paying attention to the wrong things. What else might I hear if I listen?

The next day Greg returns to the table where Jerome and Malik are continuing to find more boxes. Malik is describing his new strategy to a skeptical Jerome. "I think that we can use this big box to make even bigger boxes." He is referring to an 8 × 5 box they have made with four smaller boxes, each 2 × 5. See figure 4.2. Malik continues, "We don't have to always use the little ones [pointing to the 2 × 5s] because they're already in there. Look, if we put two of these together, we know that it's 80 because it's two 40s. There's eight 2 × 5s in there because there's four 2 × 5s in each of these two boxes."

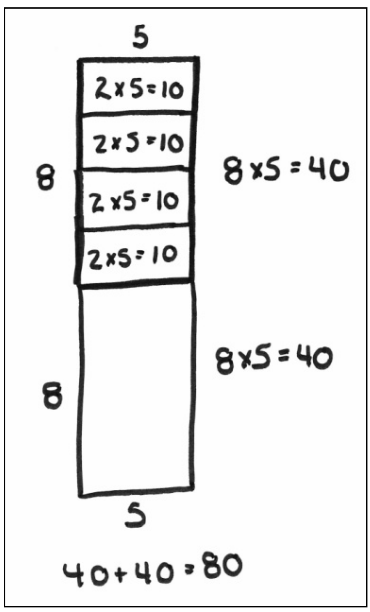

Fig. 4.2. An 8 × 5 box containing four smaller, 2 × 5 boxes

Greg is impressed again, but he wonders whether Malik can figure this out only because the numbers are multiples of five and ten. Will Malik be back to drawings with many squares if asked to think about other numbers? Greg decides to check. “That is really interesting, Malik. Do you think that strategy could work with boxes other than 2 by 5? What about 2-by-6 boxes? What larger boxes could you make with those?”

Malik pauses to think and then draws outlines of four boxes and labels each one “2 × 6.” “That’s 12 in each one … 12, 24, 36, … ummm … 48. That means two big boxes is … 48 plus 48, that’s …” Malik’s enthusiasm wanes.

Realizing that the numbers have gotten too big for Malik to think about easily, Greg writes “48 + 48” and helps him decompose the numbers. “Do you know what 40 + 40 is? And 8 + 8?”

Malik beams, “80 + 16 … it’s 96! It works! Eight 2 by 6s makes 96. I built it with two 8 by 6s!” Figure 4.3 shows Malik’s work.

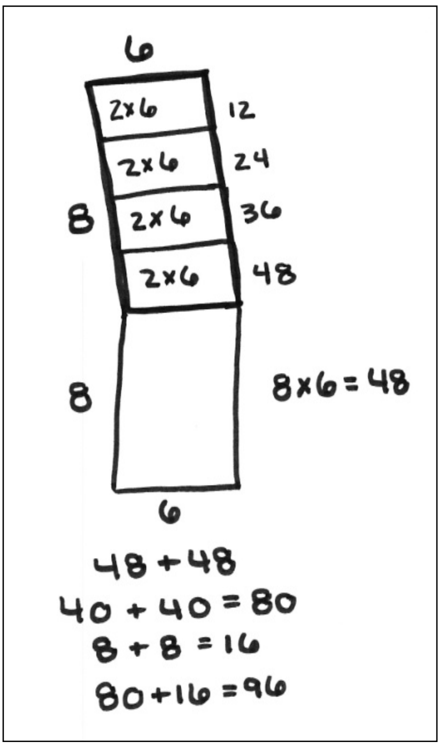

Fig. 4.3. Malik’s work with 2 × 6 boxes

Greg celebrates with Malik his accomplishment and then encourages him to try working with 2 × 3 boxes as well. Malik outlines four more boxes, labeling each one as 2 × 3. “That’s 6 in each box, so the 4 by 6 is 12 plus

12. If I put two of these together, that's … ummm … 24, and 24 doubled, that's 48. Look at this." Pointing to his work (fig. 4.4), he declares with confidence, "It's eight sixes instead of eight tens, and so it's 48, not 80."

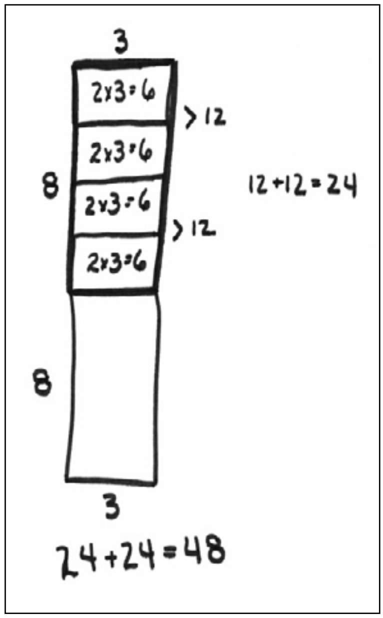

Fig. 4.4. Malik's work with 2 × 3 boxes

Greg leaves the boys to continue investigating. He knows that 8 × 6 is a fact that Malik does not know automatically. But the doubling Malik is doing helps him derive the products quickly, even though he may not yet recognize the full importance of his discovery: how doubling and halving generalizes to the associative property. Greg leaves with evidence that Malik understands much more than he had previously thought, and he has a clear sense of direction of how to support Malik's development in the days to come. That night in his journal, he reflects on his questioning:

> When I asked a question, it was not to get Malik closer to finishing the work so that he would be ready to share with the class. Nor was it about trying to lead him to my answer. It was because I was genuinely curious about what he was thinking. The powerful questions I asked elicited more information about how deep his understanding about his new idea really was. I will look for more opportunities for my students to work in partnerships on problems about big ideas, spend more time eavesdropping as I walk around, and try to wait to ask more questions about thinking that I'm genuinely curious about.

When partnerships go awry

When teachers come to a group at work, it is helpful not only to assess each student's progress individually but also to observe whether the pairing or grouping is contributing to, or hindering, learning. The tendency is to place a strong math student with a struggling student, but teachers must undertake this pairing carefully, with reasonable goals in mind (e.g., activities in which both can be equal contributors, collecting data, building geometric figures); otherwise, the pairing is usually not productive. The stronger student spends most of his time trying to explain something he already understands, which does not challenge his math development, and this configuration reinforces the "learned helplessness" of the other as he struggles to understand what his partner is trying to explain.

Homogeneous grouping can be useful if both students are working on particular strategies or similar ideas, or rehearsing to present their ideas to the class, but such a format can also be limited in exposing the students to a diversity of ideas. Research by Doise and Mugny (1984) shows that the most powerful pairing is an "optimal mismatch": a social interaction that promotes puzzlement, usually comprising one person who is on the cusp of constructing an important idea and one who has been trying to use it but is not solid yet in the understanding of it. Here, the stronger student's attempts to explain his thinking to the weaker student help him clarify his own thinking. The other student benefits also because the talk challenges him and is in his zone of proximal development (Vygotsky 1978).

Of course, in the real world of the classroom we cannot always choose such perfect combinations every time, and so often adjusting in the moment may be needed. Sometimes after listening to and observing the group interaction, a teacher may decide to participate along with the students to challenge them. For example, students in another fourth-grade class were playing a game in which they were discussing multiplication strategies. The teacher observed that in one group students were getting along well and were engaged. But after listening to their conversation she found that they were guessing the answers to the multiplication problems, whereas in other groups students were helping each other by breaking down factors into workable partial products, using the distributive property, and using arrays as visual models. (An array is an area model for multiplication that consists of an arrangement of objects, pictures, or numbers in rows and columns. A typical egg carton, for example, is a 2×6 array. The cards the children were using were made from one-inch graph paper, arrays of squares in rows and columns.) The teacher intervened with the first group, first reviewing the purpose of the game and then actively playing the game with them. She modeled how she might solve 4×8 by using 4×4 as a starting point and showing them the partial product,

4×4, in relation to 4×8 by covering the partial area. As the students each then took a turn, she noted the arrays they used. Observing their interactions as they played together gave her insight into their thinking that she might not have learned from watching each separately or studying individual pieces of work.

At other times, a teacher may decide that a particular pairing is not meeting the needs of the students and that she needs to think about other arrangements. Two kindergartners were doing an activity in which they were supposed to grab two handfuls of cubes, line them up, compare them, and color the corresponding numbers on paper strips. This activity was designed to support the development of cardinality, one-to-one correspondence, and magnitude (comparing the quantities). The teacher noticed that for one pair the activity had become only a coloring exercise with no apparent one-to-one correspondence or comparing. She decided to play a few rounds with them, asking the children how many cubes she had for each round. She succeeded in focusing the attention of one child, but the other continued coloring. The teacher realized that this pairing was not working. She decided to pair the first child with another student who already had a good strategy for comparing and recording and to do some one-on-one work with the second child, playing the game with her and focusing on the comparing instead of the coloring.

Keeping track

The roles teachers take as listeners vary according to the circumstances. Sometimes they just listen and take notes either on the spot or as soon as possible after the interaction. For ongoing assessment, teachers must record what they notice in these interactions. Teachers find their own method of recording that works for them, using a clipboard, sticky notes, or other systems. Keeping track of what and how students are learning during group activities and games can be especially difficult.

One teacher recorded how children were playing a card game, Close to 100 (Russell et al. 2008b), in which players draw six cards from a set of digit cards (one through nine) and choose four to make two two-digit numbers that when combined will result in a sum as close to 100 as possible. For example, if a hand consists of 3, 6, 5, 1, 4, 9, among the best choices would be 5, 1, 4, and 9 (to make 51 and 49, or 59 and 41, which add up to 100), or 3, 6, 4, and 5 (to make 64 and 35, or 34 and 65, which add up to 99). She noted what strategies they were using and how they worked with partners on a chart:

Melissa	No problem combining numbers but appears to be picking cards randomly; partner is trying to explain her strategies for picking cards but not finding the best words
Katrina	Found good combinations but struggled to explain to her partner why she had the best combination (partner was just picking numbers randomly)
Sara	(Worked with before the game) Miscalculated the difference from 100 but choosing good cards
Deneisha	Good thinking ahead; she looks for numbers that can make nines and tens as a way to make numbers that equal 100; had to be prompted for equations

Since classes often play games repeatedly over time, the teacher could refer to her notes before the class played the game again. She looked for students who needed more support and decided how to pair the students. For example, she changed the preceding pairs around so that Melissa, instead of working with Katrina, now worked with Sara, a student who made computational errors but was choosing good cards. Then she went through the same process, taking notes about what students were saying and doing during the game, and evaluating both their individual progress and the effect of the different groups.

Other teachers have run off multiple copies of the landscape graphics from chapter 2. Using anecdotal notes and samples of children's work as evidence, they highlight the landmarks on the landscape as children show evidence of them, thereby providing a trace of the developmental journey.

Conferring One on One

Capturing where the child is on the landscape of learning—where she has been, what her struggles are, and where she is going—can be difficult in the heart of teaching. Children do not always speak about what they do not know. They have worked hard to solve a problem and their answer makes sense to them. They may be unaware that their answer is incorrect and that there is something they do not understand. Children also do not always speak about the many things they do know. Perhaps they think it is not necessary to say anything, assuming that if they know, surely their teacher must know too. Perhaps they do not share their ideas because things that they know are no longer a puzzle and so are no longer interesting to them. Perhaps they have never had the opportunity to describe their knowing to someone who could give a name to their ideas and celebrate the mathematical importance of each one and so make them seem like something worth sharing.

Conferring one on one with a child may sometimes be necessary to understand better what he or she knows. Children can, and often do, speak about what they are doing to figure oust a problem and what they are puzzling over. Figuring out what children are struggling with can sometimes be as simple as finding opportunities to listen and confer one on one with them as they try to make sense of a problem.

Heather is a fourth grader, a compliant student who works hard but who struggles in math class. She depends on her partner or on an adult to get started. She is also self-conscious about her difficulties and rarely speaks in class. Lauren, her teacher, decides to confer with her one on one to get a better sense of what Heather knows and what her challenges are. In preparation, she asks Heather to work on some two-digit-by-two-digit multiplication problems. To confer, Lauren beckons her to a quiet section of the classroom and notes that she has completed some problems but has not solved any of them correctly. Heather has started off using the traditional algorithm, and her paper is scarred with many erasures and crossouts. Farther down the page Heather apparently had abandoned the algorithm in favor of another strategy that resulted in far fewer crossouts and eraser marks, but unfortunately still no correct answers. She has used a mixture of addition and multiplication, and at first glance Lauren sees no consistent process. Figure 4.5 shows Heather's work.

Lauren is curious why Heather shifted from the algorithm that she was excited about using and had been practicing at home with her mom. She also wonders where Heather found this new strategy. "Heather, it looks like you have been working really hard. Can you tell me a little bit about what you've been doing?" Lauren says warmly.

Heather points at the crossouts on her paper and says, "I was trying to use the way my mom showed me, but I keep forgetting how it goes. It has lots of steps, and sometimes I get confused."

Lauren points to where Heather had started using the second strategy and asks, "Can you tell me a little bit about what you're doing down here?"

Heather answers, "Well, I didn't know what to do, so I thought about Shameeka's strategy. She was using tens and I know tens are friendly numbers, so I thought I would use tens."

"That's great that you tried to find a way to make sense of the problems for yourself. Tens sometimes can be very helpful." Lauren encourages her

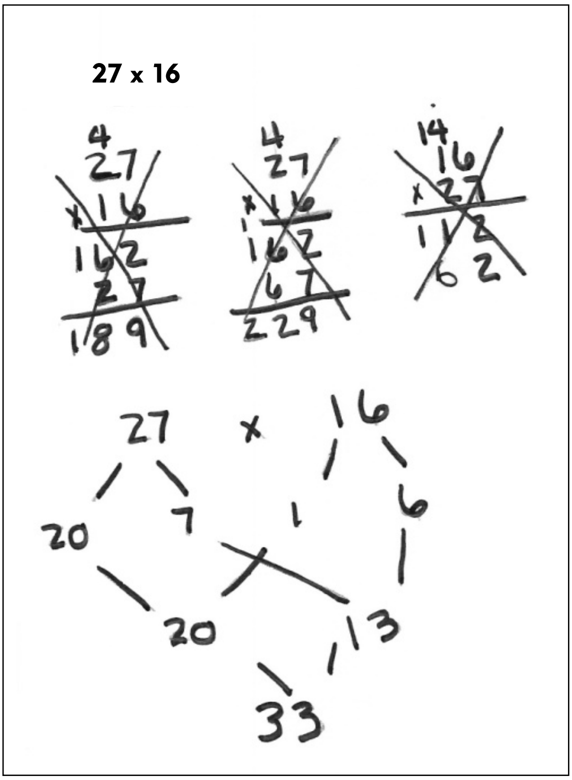

Fig. 4.5. Heather's work, multiplying 27 × 16

to trust in her own sense making while continuing to probe: "Show me how you used tens to help."

Heather describes her thinking to Lauren. "I knew that you had to separate the tens and the ones, so I took the tens out and I put them together, and then I took the ones out and put them together, and then I put them all together." She gets stuck when she gets to the 1 recorded below the 16. She starts again with the 27, but her face falls and she pauses in confusion when she gets back to the 16.

"So you took a ten out of 27?" asks Lauren, hoping to prompt reflection by responding with a very literal interpretation of Heather's statement.

"No, I took 20 out of 27 and a ten out of 16," Heather replies.

Lauren records the 20 below the 27 and a 10 below the 16. "Oh, okay. Does that look right so far? It is different from what you wrote before, and I want to make sure I am recording your thinking exactly," Lauren tells her.

"Yes, that's what I did. I guess I was rushing before; I should have been neater," Heather replies, eyes dry but lips still trembling slightly.

Lauren acknowledges Heather's comment with a smile and a nod, hoping to encourage her, and returns the focus to the problem. "Okay, I think I have it now. What is the next thing you did?"

"I took the ones out," Heather responds, this time with a little more confidence.

Lauren celebrates that Heather knows that the 2 in 27 and the 1 in 16 represent the number of tens. "So we have 20 + 7 and 10 + 6. You know a lot about these numbers." Heather beams and Lauren continues, "Now how do we do the multiplication?"

"You do twenty times ten, and that equals 200. Next to it you need to write 7 × 6 =...." There is a long pause and then, "Oh, I don't know 7 times 6," Heather sighs.

The most important thing to do when a learner is stuck is to identify what the learner does know and then build on it. Lauren asks, "Are there any 7 times that you do know?"

Heather responds, "I know 2 × 7 = 14 and 3 × 7 = 21, but that's all."

Lauren wonders aloud, "Hmm, do you think those 7-times facts could help us figure out 7 × 6?"

"I don't think so," Heather answers hesitantly.

Lauren makes a note that Heather does have some known facts in her repertoire but does not know how to use them to figure out other known facts. "Is there any way we could figure out 7 × 6 so we could finish this problem?"

Heather responds, "Well, I could add 7 six times."

Lauren has now elicited another important fact about what Heather knows: Heather thinks about multiplication as repeated addition. Lauren begins there. "Let's try that and see if it helps us." Lauren records "7 + 7 + 7 + 7 + 7 + 7" off to the side of the paper, to the right of her previous thinking. "What should I write now?"

This time Heather responds quickly, "I know double seven is fourteen, so it is three fourteens." Another landmark on the landscape of learning is now apparent: Heather can regroup the groups. Lauren records her thinking as 3 × (2 × 7) and celebrates this achievement. "Wow. That's a good way to think about it. So what should I write next?"

Heather replies, "14 + 14 equals 28, and then you just add the last 14 and that's 32, no, 42!" She flashes a big grin.

"Great! So now we know 7 × 6. What do we do next?"

Heather pauses briefly and then says, "We need to add the 200 and the 42, and that is 242. Now we are done!"

Learners often struggle so hard to understand procedures that they lose track of where they are in a problem. Lauren wants to encourage Heather to rely on sense making rather than on procedures. Toward this aim, Lauren encourages her to examine her answer: "Do you think 242 can be right? Is it big enough?"

Heather pauses, and her eyes begin to well up again. "I don't think this can be right either. I think it's too little. It's like if you had quarters. Four quarters makes a dollar, so eight quarters makes two dollars, and if you double that then you get four dollars. So that's 400 and even if you take some pennies away it's still bigger than 242."

In this conference, Lauren has succeeded thus far in establishing several things about where Heather is on the landscape. Heather knows that ten is an important number in our number system. She can use place value to decompose numbers. She recognizes the connection between multiplication and addition. Most important, she can use doubling as a strategy and come up with a strategy for solving a problem when she is not constrained to use a procedure she does not understand. She can regroup the groups to make partial products with known facts when the repeated addition is present, but her use of partial products is not flexible. Earlier, although she knew the partial products of 2 × 7 and 3 × 7, she could not envision how they might help to figure out 7 × 6. Using partial products with larger numbers is also a hurdle for her. She makes two of the necessary partial products but is missing the other two. Heather also told Lauren something that she could not do and why it was hard for her: the multiple steps in the standard algorithm were problematic because she could not remember them all, even though she practiced them at home every night.

Lauren decides to build on what Heather does know and challenge her to use this understanding with larger numbers, hoping also to help her build some meaning for the standard algorithm she is trying to use. She gets out graph paper. "Let's draw a rectangle that has 16 rows of 27," she begins. "Let's see if we can find the pieces you did." On the side she writes "27" sixteen times to show the repeated addition. "How could we begin?"

Heather begins with the doubling she is comfortable with. "I know 27 + 27. That's 54." Lauren outlines a 2 × 27 array and continues to do more of these as Heather describes that eight of these fill the array. Together they arrive at the solution of 432, by adding 54 eight times. See figure 4.6. Heather is beaming.

"Wow … that way worked," Heather says. Lauren smiles back at her and then challenges her to find another way. "Do you know what 10 × 27 would be?" Lauren outlines a new 16 × 27 array, and this time she traces over the partial product of 10 × 27.

"Yep, I know that, too. It's 270," Heather declares with increasing confidence.

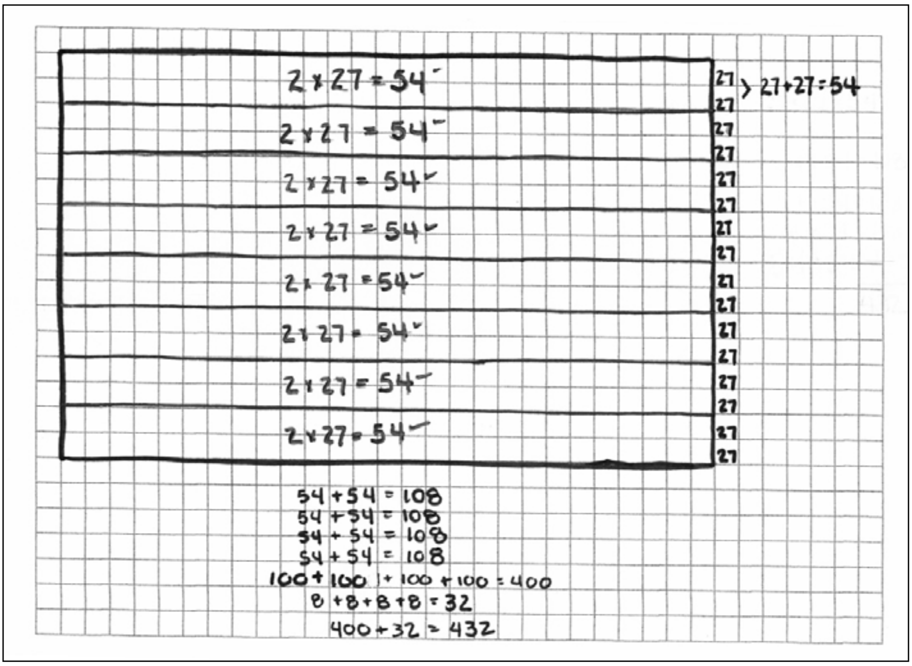

Fig. 4.6. One way to solve 16 × 27

"Okay, let's see what we have left. Here's the 6 × 7 that you figured out was 42. Let's draw that in. What's left?" Lauren asks.

"Six rows of 20. I know 20 + 20, that's 40. I need three of them … that's … 120," Heather responds.

Lauren completes the work shown in figure 4.7, and Heather adds 270 + 42 + 120, declaring that she got 432 again.

"Wow … now you know two ways to do this!" Once again Lauren celebrates and then challenges, "Do you think we could find the 20 × 10 that you did before?"

Doing so proves difficult, but Lauren helps her find it and then they use the picture to examine the two partial products that Heather had missed initially (20 × 6 and 7 × 10). See figure 4.8. Lauren also uses the picture to examine how the standard algorithm uses these same partial products (6 × 7 + 6 × 20 + 10 × 7 + 10 × 20), albeit in the reverse order from what Heather used. During the one-on-one conferring, Lauren has learned a lot about how to help Heather.

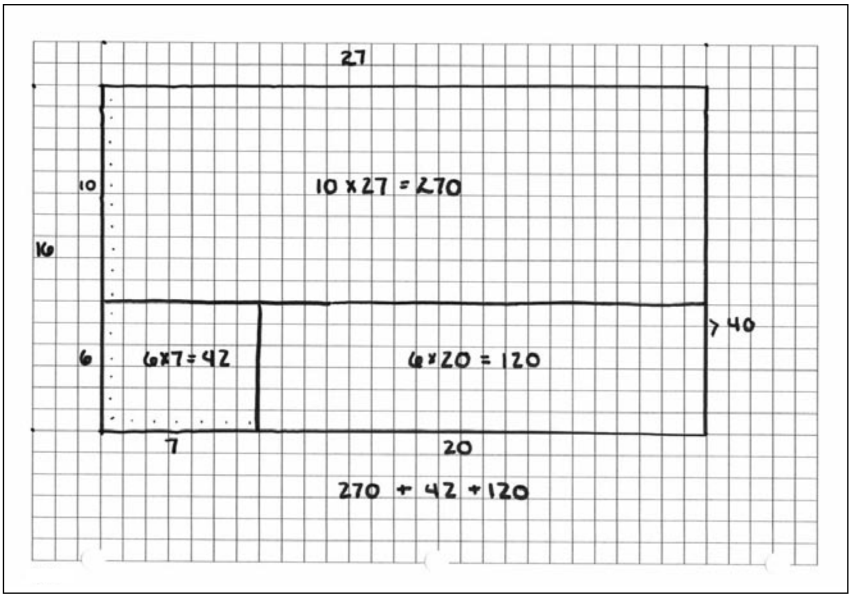

Fig. 4.7. Another way to solve 16 × 27

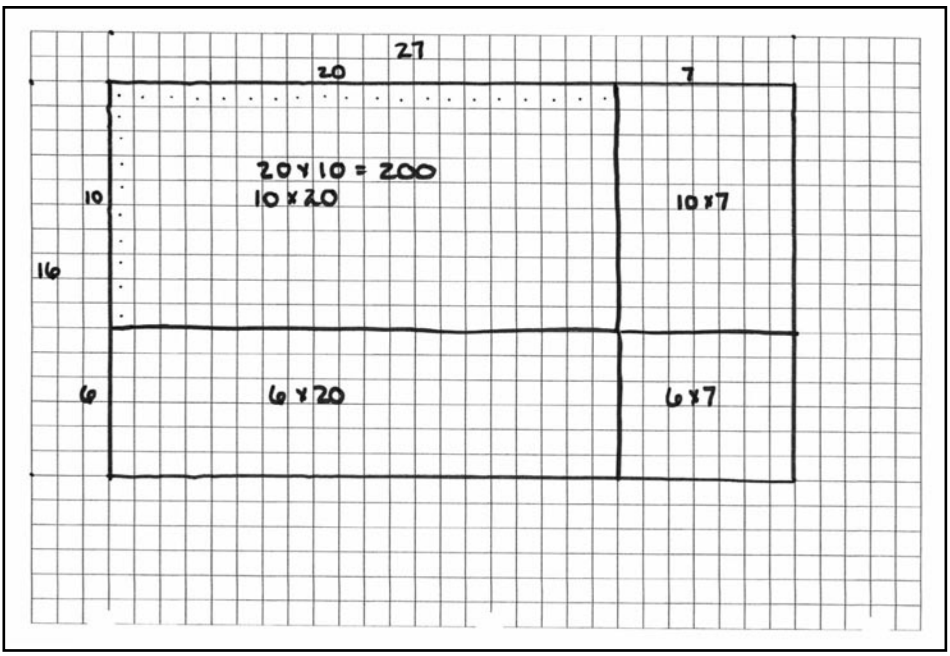

Fig. 4.8. Partial products used to solve 16 × 27

Capturing Genuine Mathematizing Formally

Time does not permit the luxury of sitting with each student each day, so sometimes using more formal tools of assessment is helpful to determine what individual children know. There is a difference between writing about how you solved a problem and having the work visible. Too often, children are told, "Write in words, pictures, and symbols what you did." Sometimes they have actually done the problem mentally, but to please the teacher they describe a more inefficient strategy. Such statements on assessments can also confound literacy with mathematics, placing ELL (English language learning) students at a disadvantage to communicate what they really know. If we want to capture actual thinking, supplying scrap paper as part of the test is more helpful (van den Heuvel-Panhuizen 1996). Further, requiring students to use pens rather than pencils guarantees that all marks stay visible—nothing can be erased: the paper captures different starts, changes in strategies, mistakes, rewriting, and final figuring. For example, note the three solutions in figures 4.9–11.

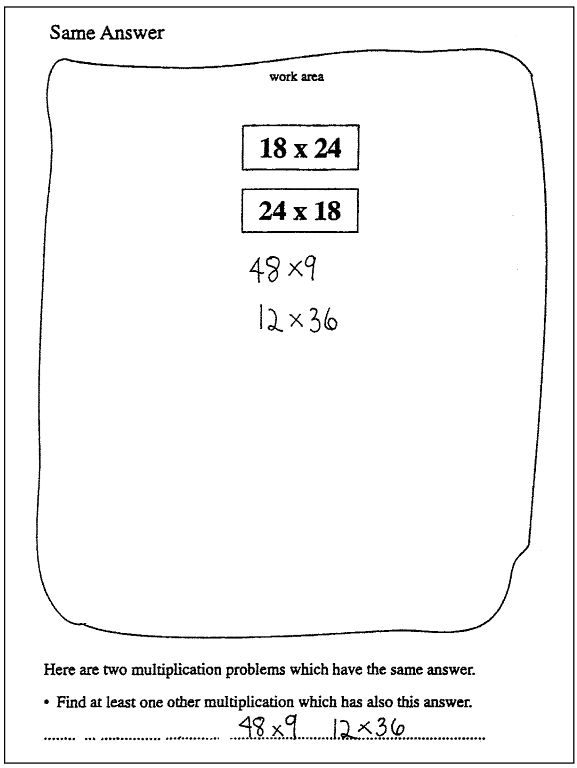

Fig. 4.9. The first child solves the problem mentally with doubling and halving procedures, showing evidence of understanding the associative property: (24 × 2) × 9 = 24 × (2 × 9).

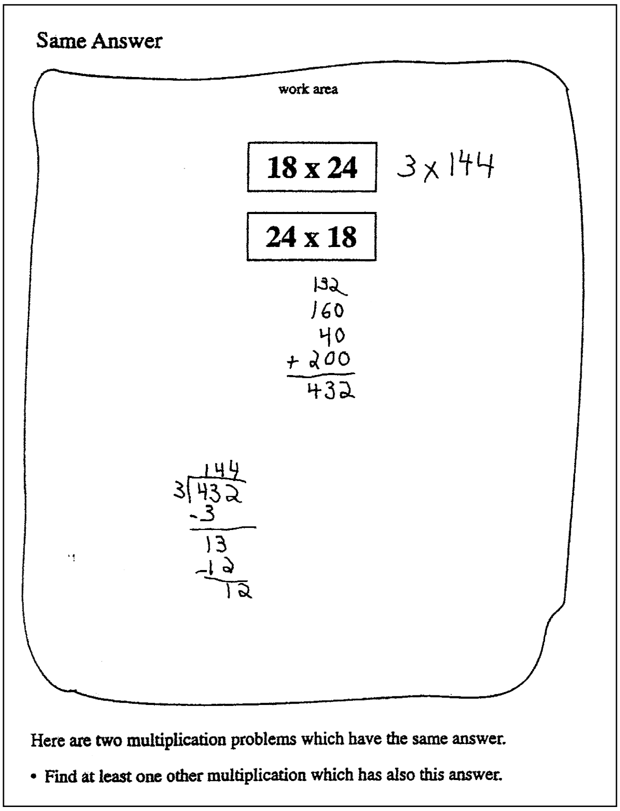

Fig. 4.10. The second child succeeds but in a more tedious fashion. She does all the multiplication first by adding up the partial products and then uses long division, choosing 3 as a divisor (possibly even a lucky choice). Although this work shows evidence of an understanding of the connection between multiplication and division, it does not show knowledge of other, more efficient strategies to find equivalent expressions.

Comparing the visible mathematizing of these three children gives the teacher much usable information about where they are on the landscape of learning and the landmarks and horizons ahead.

Two-pen assessments can also yield information (van den Heuvel-Panhuizen 1996). Consider the assessment in figure 4.12. The problems are related, chosen to capture children's understanding of the relations and to pick up information on their strategies. Children use a red pen for the first two minutes and first do all the problems they see that are easy for them. Then they switch colors and complete the assessment. If children find 6×7 easy, why do they not find 60×70 easy? Often children will use the standard algorithm for 60×70, have trouble with all the zeros, and make place-value errors rather than just seeing it as $100 \times 6 \times 7$. If they find $100 \div 4$ easy but do not find $300 \div 12$ easy, then they do not see how

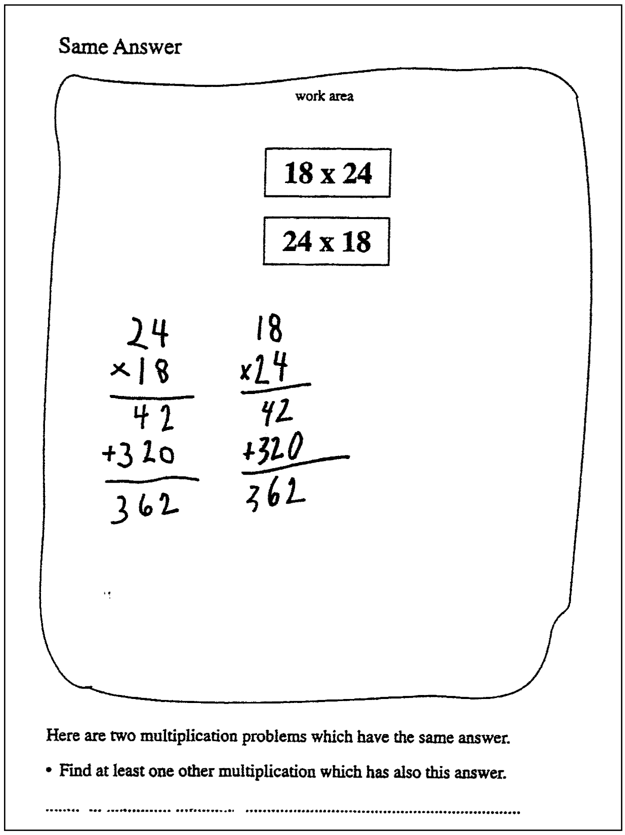

Fig. 4.11. The third child also attempts to find the product first and makes errors with the algorithm.

Name ______________________ Date ____________

6×7	60×70
$396 \div 36$	$260 \div 20$
$100 \div 4$	$300 \div 12$
$140 \div 14$	41×12
4×100	16×25
$168 \div 14$	$260 \div 13$

Fig. 4.12. Two-pen assessment (this sample assessment item was developed as part of a DYO [Design Your Own Assessment] project, a collaboration between the New York City Department of Education and Mathematics in the City)

these expressions are equivalent, how simplifying a division problem first can make it easy. Examine the relationships of the problems in the figure. Using this item, teachers can assess many other ideas as well.

Linking assessment contexts to reality and providing for various levels of mathematizing

Using realistic contexts can also help to capture children's genuine mathematizing. They must be more than word problems camouflaging "school mathematics." They must be real or be able to be imagined by children. One approach is using pictures or telling stories (see fig. 4.13).

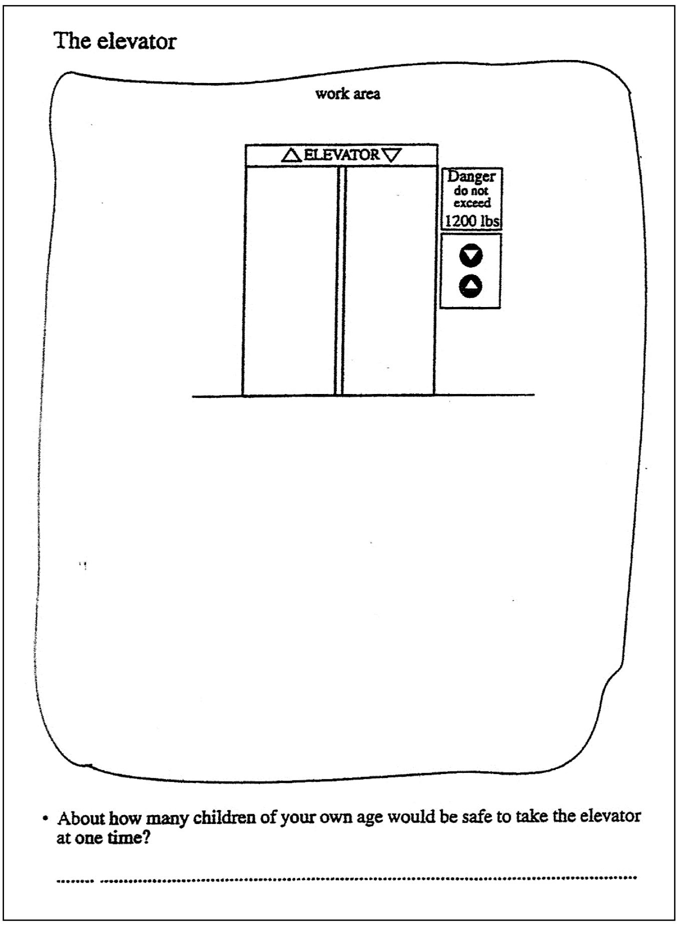

Fig. 4.13. The elevator problem

Opening up assessment by using realistic situations lets teachers look at *how* students find the answer to the question, not just *whether* they find it. For example, many ways exist to solve the "elevator" problem in figure

4.13. Children capable of a high level of mathematizing would choose a reasonable estimate of the weight of a fifth grader (a friendly number related to 12, such as 60 or 80) and then reduce both the dividend and the divisor: 1200 divided by 80 is equivalent to 300 divided by 20, which is equivalent to 30 divided by 2, which is 15. Other children might ask each classmate his or her weight, add them all up with an algorithm, use the long division algorithm to find the average, and then use the algorithm once again to divide this average weight into 1200. Still others might proceed randomly, trying to add various weights to reach 1200: a much lower level of mathematizing.

By assessing how a student mathematizes, teachers acquire information that guides their teaching. They can understand where the child is on the landscape of learning. By analyzing the child's markings on the paper given with the test, teachers can comprehend not only how the child is currently mathematizing but also what strategies she is trying out. Because the landmarks on the landscape of learning become visible, teachers can determine appropriate horizons. Knowledge of the landmarks the students pass (collectively and individually) in their journey through this landscape shapes teachers' questions, their instructional decisions, and the curriculum. In this way, learning and teaching are connected. When the primary function of assessment is to influence teaching, evaluation of learning is also redefined. Rather than "grading" children with scores, we can document the developmental journey.

Documenting the Journey

Teachers have different ways to keep track of children's journeys. Some teachers copy the figures in chapter 2 and use them as a graphic representation of each child's journey. They highlight each landmark as children reach them, producing a trail of the development. As evidence of the trail, they also make copies of children's work and include their anecdotal records, as well as results of assessments such as those described here. Teachers collect these items in a folder, which serves as important evidence of learning as teachers confer with parents, discuss individualized education programs, and collaborate with colleagues.

The pathways along this journey are not necessarily sequential. Teachers and students can take many paths toward this horizon. Some landmarks are, of course, precursors to others: repeated addition is a precursor to unitizing and the distributive property. Other children will develop computation strategies that work, such as doubling and halving, before fully understanding why they work: they try out the strategy and only later construct the big idea that the associative property explains the strategy. Still others will construct the big idea first. The landmarks are not a checklist or a list of behavioral outcomes; they are a means to focus on and describe students' mathematics development. They represent the cognitive reorganization learners make as they journey toward becoming competent mathematicians.

(We gratefully acknowledge the contributions of Michael Flynn, Maureen McCarty, Mary Kay Archer, Lisa Seyferth, Nikki Faria, and Ana Vaisenstein. Portions of some of their stories in this chapter were previously published in a book edited by Judy Storeygard and are published here with their permission.)

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